En el espacio vectorial $P_2(\mathbb{R})$ de los polinomios de grado menor o igual a 2 con coeficientes en $\mathbb{R}$, consideramos el subespacio vectorial

$$U = \{p(x) \,/\, p(0) = p''(1)\}.$$

Se pide:

a) Calcular B_U una base y la dimensión de U.

b) ¿Pertenece al subespacio U el polinomio $p(x) = x^2 - x + 1$? En caso afirmativo, calcular las coordenadas de $p(x)$ en la base obtenida en el apartado anterior.

c) Consideremos en $P_2(\mathbb{R})$ el producto escalar cuya matriz de Gram respecto de cierta base $B=\{p_1, p_2, p_3\}$ es:

$$G = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Se pide:

c.1. ¿Son perpendiculares los polinomios p_1 y p_2? ¿Es el vector p_3 unitario? Razonar las respuestas.

c.2. Calcular el ángulo que forman los vectores $p(x) = p_1 + p_2$ y $q(x) = p_1 - p_3$.

c.3. ¿Es B una base ortogonal? En caso negativo, calcular una base ortonormal.

$$U = \{\underbrace{a + bx + cx^2}_{p(x)} \,/\, p(0) = p''(1)\}$$

$p(0) = a$

$p'(x) = b + 2cx \Rightarrow p''(x) = 2c \Rightarrow p''(1) = 2c$

$U = \{a + bx + cx^2 \,/\, a = 2c\}$

$a - 2c = 0 \Rightarrow \left.\begin{array}{l} a = 2\lambda \\ b = \mu \\ c = \lambda \end{array}\right\}$ Ec. paramétricas

$B_U = \{(2,0,1), (0,1,0)\}$ base de U

$\{2 + x^2, x\}$

$(2,0,1) \equiv 2 \cdot 1 + 0 \cdot x + 1 \cdot x^2 = 2 + x^2$

$(0,1,0) \equiv 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 = x$

b)

$p(x) = x^2 - x + 1$

$p(0) = 1 \neq p''(1) = 2$

$\underline{p(x) \notin U}$

¿$\exists\, \alpha, \beta \in \mathbb{R}$ tq $p(x) = \alpha(2 + x^2) + \beta \cdot x$

$x^2 - x + 1 = \alpha x^2 + \beta x + 2\alpha$

S. Incomp. $\alpha = 1$, $\beta = -1$ $2\alpha = 1$

c) $B = \{P_1, P_2, P_3\}$

$$G = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

c.1) $P_1 \perp P_2 \iff \underbrace{< P_1, P_2 >}_{a_{12} \text{ de la matriz de Gram}} = 0$

P_3 es unitario $\iff \|P_3\| = 1$

$$\|P_3\| = \sqrt{< P_3, P_3 >} \Rightarrow < P_3, P_3 > = 1 \quad \#$$

$$< P_3, P_3 > = a_{33} = 2$$

c.2) $p(x) = P_1 + P_2 \equiv (1, 1, 0)_B$

$q(x) = P_1 - P_3 \equiv (1, 0, -1)_B$

$$\cos(p(x) \hat{,} q(x)) = \frac{< p(x), q(x) >}{\|p(x)\| . \|q(x)\|}$$

$$< p(x), q(x) > = (1 \ 1 \ 0) \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} =$$

$$= (2, 1 \ 1) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1$$

$$\| p(x) \|^2 = < p(x), p(x) > = (1 \ 1 \ 0) \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} =$$

$$= (2 \ 1 \ 1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3$$

$\Rightarrow \|p(x)\| = \sqrt{3}$

$$\| q(x) \|^2 = < q(x), q(x) > = (1 \ 0 \ -1) \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} =$$

$$= (1 \ 0 \ -1) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2 \Rightarrow \|q(x)\| = \sqrt{2}$$

$$\cos(\alpha) = \frac{1}{\sqrt{3} . \sqrt{2}} = \frac{1}{\sqrt{6}} \Rightarrow \alpha = \arccos\left(\frac{1}{\sqrt{6}}\right)$$

c.3) B no es ortogonal $\Longleftrightarrow$ G no es diagonal

$$a_{13} = \langle P_1, P_3 \rangle \neq 0$$

$B_{ortg} = \{u_1, u_2, u_3\} \rightsquigarrow$ Gram-Schmidt

$$u_1 = P_1 \equiv (1,0,0)_B$$

$$u_2 = P_2 - \frac{\overbrace{\langle P_1, P_2 \rangle}^{=0}}{\|P_1\|^2} \cdot P_1 = P_2 \equiv (0,1,0)_B$$

$$u_3 = P_3 - \frac{\langle P_3, P_1 \rangle}{\|P_1\|^2} \cdot P_1 - \cancel{\frac{\langle P_3, P_2 \rangle^{=0}}{\|P_2\|^2}} \cdot P_2 =$$

$$= P_3 - \frac{1}{2} P_1 = -\frac{1}{2} P_1 + P_3 \equiv \left(-\frac{1}{2}, 0, 1\right)_B$$

$$B_{ortg} = \left\{ (1,0,0), (0,1,0), \left(-\tfrac{1}{2}, 0, 1\right) \right\}$$

$$B_{ortn} = \left\{ \frac{(1,0,0)}{\underbrace{\|(1,0,0)\|}_{\sqrt{2}}}, \frac{(0,1,0)}{\underbrace{\|(0,1,0)\|}_{1}}, \frac{\left(-\frac{1}{2},0,1\right)}{\left\|\left(-\frac{1}{2},0,1\right)\right\|} \right\}$$

$$\left\|\left(-\tfrac{1}{2},0,1\right)\right\|^2 = \left(-\tfrac{1}{2}, 0, 1\right) \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} =$$

$$= \left(0, 0, \tfrac{3}{2}\right) \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} = \frac{3}{2} \Rightarrow \left\|\left(-\tfrac{1}{2},0,1\right)\right\| = \sqrt{\tfrac{3}{2}}$$

$$B_{ortn} = \left\{ \left(\frac{1}{\sqrt{2}}, 0, 0\right), (0,1,0), \left(\frac{-1/2}{\sqrt{3/2}}, 0, \frac{1}{\sqrt{3/2}}\right) \right\}$$

$$= \left\{ \left(\frac{1}{\sqrt{2}}, 0, 0\right), (0,1,0), \left(-\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{\sqrt{2}}{\sqrt{3}}\right) \right\}$$